Rafael Bombelli's L'Algebra

Daniel J. Curtin[*]
Northern Kentucky University

C. Edward Sandifer[y]
Western Connecticut State University

Gary S. Stoudt[z]
Indiana University of Pennsylvania

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1 Introduction

Thanks to the Institute for the History of Mathematics and Its Use in Teaching (IHMT), the authors have developed a (perhaps unusual) fondness for the work of the Italian mathematician Rafael Bombelli. We were assigned to find out what was really in his work L'Algebra and thanks to the wonderful Artemas Martin Collection in the library at American University, we were able to begin the task.

[*]Department of Mathematics and Computer Science, Northern Kentucky University, Highland Heights, KY 41099-5377 U.S.A (curtin@nku.edu)

[y]Department of Mathematics and Computer Science, Western Connecticut State University, Danbury, CT 06810 U.S.A (sandifer@wcsu.ctstateu.edu)

[z]Department of Mathematics, Indiana University of Pennsylvania, Indiana, PA 15705-1072 U.S.A (gsstoudt@grove.iup.edu)

Remark 1 Modern notation is intermingled with Bombelli's notation. The reader should have no di±culty distinguishing between the two.

2 General Information

The text is in good shape. The table of contents is quite detailed, consisting of 40 pages. The text itself is 650 pages with four pages of errata. Book I is on pages 1 to 199, Book II is on pages 201 to 411, Book III is on pages 412 to 650.

3 Book I

On pages 1 and 2 we have some interesting de¯nitions:

- De¯nition of a square number (\quadrato")
- De¯nition of a cube number (\cubo")
- De¯nition of a fourth power (\quadroquadrato")
- De¯nition of a ¯fth power (\numero detto primo relato").
- A product of a square number and a cube number on the same side will be a number of \primo relato" or \primo incomposto".
- De¯nition of a sixth power (\quadrocubico" or \cubicoquarto")
- A cube number multiplied by itself.

On pages 3, 4, and 5 we have the de¯nition of \radice quadrata dette sorda overo indiscreta", the side of a number that is not quadrato. For example, $\sqrt{20}$ is discussed, but not calculated.

On page 4, the notation R.q. for square root is introduced. Also on pages 4 and 5, the notations for cube root (R.c.), fourth root (RR.q), ¯fth root, or “radice prima relata" or “radice incomposta" (R.p.r), and the sixth root or “radice quadracubica" (R.q.c) are given. On page 6, a summary of notation is given, including seventh root or “radice seconda relata" or “radice seconda incomposta" (R.s.r)

Next we have a Note (Avertimenti). “A square number multiplied by a square number is a square number." Also, “a square number multiplied by an unsquare number is an unsquare number." An example follows.

On pages 8 and 9 we see the multiplication of a radical with a number. The example given is $7 \times \sqrt{5}$ which is “solved" by writing $\sqrt{49} \times \sqrt{5} = \sqrt{245}$. Note here that for notational purposes, Bombelli prefers one “radice", R.q.245, where we would have our students simplify to $7 \times \sqrt{5}$; that is, leave the multiplication by a radical with a number as it is. This is not the only time Bombelli will do this.

Next on page 10 is division (“paratasi") of a radical by a number. The example given is $\sqrt{7} \div 5$ which Bombelli writes as $\sqrt{7} \div \sqrt{25} = \sqrt{\frac{7}{25}}$ or R.q.7 partire 25.

This problem is followed by adding a square root and a number. Bombelli explains that you cannot compute this, you just write it (non si può fare). For example, $4 + \sqrt{7}$ is R.q.7.p.4. On page 11 Bombelli explains that when writing this, the larger number goes first. We would write $4 + \sqrt{20}$, Bombelli writes R.q.20.p.4; we write $6 + \sqrt{2}$ and Bombelli writes 6 p.R.q.2.

On pages 11 and 12 Bombelli discusses the sum of roots and roots, which is, as he says, “più difficile". He writes the operation out in table form.

	$\sqrt{15}$		$\sqrt{135}$
	135		$\sqrt{15}$
Summa	150		$\sqrt{2025}$
		late	45
Summa	$\sqrt{240}$		2
			90

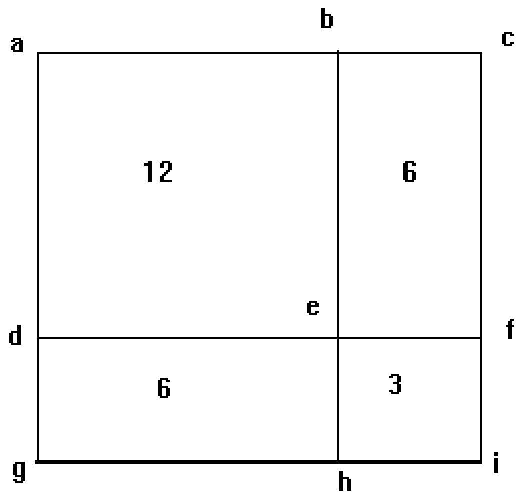

Figure 1:

On page 12 we see a geometric demonstration of the operation with the sum $\sqrt{12}+\sqrt{3}$ used (see Figure 1).

In Figure 1 abde is a square of side $\sqrt{12}$ and efhi is a square of side $\sqrt{3}$. The rectangles bcef and dgeh have sides $\sqrt{12}$ and $\sqrt{3}$ (area of 6). What Bombelli does is square $(\sqrt{12}+\sqrt{3})$, combine terms, and then take the square root. As one can see in Figure 1, $(\sqrt{12}+\sqrt{3})^2$ is 27 and therefore $(\sqrt{12}+\sqrt{3})$ is $\sqrt{27}$.

Perhaps Bombelli's calculation is now made clear:

	$\sqrt{15}$		$\sqrt{135}$
	135		$\sqrt{15}$
Summa	150		$\sqrt{2025}$
		late	45
Summa	$\sqrt{240}$		2
			90

The 135 is from a square with side $\sqrt{135}$, to which is added ("summa") 15 from the square with side $\sqrt{15}$ and to this is added two times 45 which is $\sqrt{2025}$ (the 90) which is from the two rectangles of sides $\sqrt{135}$ and $\sqrt{15}$. This sum is 240 and the square root is then taken to complete the calculation.

On page 159 we see "method of knowing of two quantities composed of irrationals which will be the larger" (which is important for Bombelli's notation, i.e. larger comes first).

On page 168 we see a simplification:

8.p.Rad.c⌊72.p.r.q.1080.⌋p.Rad.c.⌊72.m.Rad.q.1088⌋
p.Rad.c.⌊Rad.q.4352.m.16⌋m.Rad.c.⌊Rad.q.4352.p.16⌋

which reduces to the trinomial sum

Rad.c.⌊72.p.r.q.1088.⌋p.Rad.c.⌊Rad.q.4352.m.16.⌋

That is,

$$
\begin{aligned}
&8+\sqrt[3]{72+\sqrt{1088}}+\sqrt[3]{72-\sqrt{1088}}\\
&+\sqrt[3]{\sqrt{4352}-16}-\sqrt[3]{\sqrt{4352}+16}\\
=\;&\sqrt[3]{72+\sqrt{1088}}+\sqrt[3]{\sqrt{4352}-16}
\end{aligned}
$$

On the top of page 169 we find a treasure. To appreciate this treasure, recall that in the case $x^3 = cx + d$, Cardano's formula for the solution is given in [4] as

$$x=\sqrt[3]{\frac{d}{2}+\sqrt{\left(\frac{d}{2}\right)^2-\left(\frac{c}{3}\right)^3}}+\sqrt[3]{\frac{d}{2}-\sqrt{\left(\frac{d}{2}\right)^2-\left(\frac{c}{3}\right)^3}}.$$

...Capitolo di cabo eguale ꝑ tanti, e numero, quando il cabato del ferzo delli tanti ꝑ maggiore del quadrato della meta del numero come in esso Capitolo si dimostrarꝑ, laqual forte di R.q. ha nel suo Algorismo diversa operatione dall altre, ꝑ di verso nome; perche quando il cabato del ferzo del li tanti ꝑ maggiore del quadrato della meta del numero; lo excesso loro non se puo chiamare ne più, ne meno, pero lo chiamarꝑ più di meno . . . .

We found another power of cube root much different from the others which is borne of the equation $x^3 = cx + d$, when the cube of one-third of the thing is greater than the square of the half of the number [that is, $\left(\frac{c}{3}\right)^3 > \left(\frac{d}{2}\right)^2$ as in this equation which I will show, this type of square root has its own algorithm, different from the other and it has a different name; for when the [above condition on c and d] is satisfied, the excess ought not be called positive or negative, therefore I call it più di meno when this has to be added and when it has to be subtracted I will call it meno di meno, and this operation is most exceedingly necessary ("necessarissima") . . . .

Partial translations are also in [4] and [3].

For example, $2+\sqrt{-2}$ is written 2.p.di m.2 and $2-\sqrt{-2}$ is written 2.m.di m.2. Note that solutions of cubics have not come up yet (they are in Book II); Bombelli is anticipating them, however. He will make use of this idea when he does solve cubic equations.

On page 169 is a poem describing operational rules for this new idea. (Bombelli has many such poems throughout the text; more a given later.)

Più via più di meno, fà più di meno.
Meno via più di meno, fà meno di meno.
Più via meno di meno, fà meno di meno.
Meno via meno di meno, fà più di meno.
Più di meno via più di meno, fà meno.
Più di meno via meno di meno, fà più .
Meno di meno via più di meno, fà più .
Meno di meno via di meno fà meno.

Bombelli then gives many examples of working with più di meno and meno di meno. On page 170 he states that when $2 + \sqrt{-2}$ is part of the solution (of the cubic), then $2 - \sqrt{-2}$ is also part of the solution (see Cardano's formula): “it is not possible to introduce one without it being accompanied in the binomial by the other".

If you recall that in Bombelli's notation, when adding a square root and a number, the larger is written first, you can now see that Bombelli has a problem here as he cannot decide which is larger. However, Bombelli is not thinking of più di meno and meno di meno as a radical, as he explains in the text.

To see a simple example of how Bombelli works with this notion, consider one of his first examples on page 170.

$$\begin{aligned} & 4\left(\sqrt[3]{2+\sqrt{-1}}+\sqrt[3]{2-\sqrt{-1}}\right) \\ & \sqrt[3]{64}\left(\sqrt[3]{2+\sqrt{-1}}+\sqrt[3]{2-\sqrt{-1}}\right) \\ & \sqrt[3]{128+64\sqrt{-1}}+\sqrt[3]{128-64\sqrt{-1}} \end{aligned}$$

Here $64\sqrt{-1}$ is written p.di m.64. Let's note a few things. First, this is completely consistent with his approach of multiplying a radical by a number given on page 9. Second, because Bombelli is not treating p.di m.64 as a radical, he does not have to worry about the order in which 128 and p.di m.64 are written, and thirdly, he does not move “under the radical” like he does on page 9 for the same reason,namely, p.di m.64 is not a radical.

Bombelli works on copious examples from page 170 to page 199 (the end of Book I). As one example he does

$$\left(\sqrt[3]{3+i\sqrt{2}}+\sqrt[3]{3-i\sqrt{2}}\right)\times\left(\sqrt{2}-1\right).$$

On page 173 we see

MoltiplichisiR.c.b2.m.dim.R.q.3cviaR.c.b2.m.dim.R.q.3c.

That is,

$$\left(\sqrt[3]{2-\sqrt{-3}}\right)^2.$$

On page 186 he divides p.di m. by m.di m. On page 188 he calculates $\left(3+4\sqrt{-1}\right)^3$ which is remarkable because in the process he squares the term using FOIL:

$$\left(3+4\sqrt{-1}\right)^2 = 9-16+12\sqrt{-1}+12\sqrt{-1} = -7+24\sqrt{-1}.$$

4 Book II

On page 201 Bombelli mentions Diophantus (from whom he took many problems in Book III) and mentions that in Bombelli's work \cosa" will be replaced by \tanti" (our x). He also introduces his notation for tanti: $\overset{1}{[}$: On page 204 Bombelli gives names of powers along with their notation. By considering how these are named, you can give your students insight into the power rules.

Tanti	$\overset{1}{[}$
Potenza	$\overset{2}{[}$
Cubo	$\overset{3}{[}$
Potenza di potenza	$\overset{4}{[}$
Primo relato	$\overset{5}{[}$
Potenza cuba or cubo di potenza	$\overset{6}{[}$
Secondo relato	$\overset{7}{[}$
Potenza di potenza di potenza	$\overset{8}{[}$
Cubo di cubo	$\overset{9}{[}$
Potenza del primo relato	$\overset{10}{[}$
Terzo relato	$\overset{11}{[}$
Cubo di potenza di potenza	$\overset{12}{[}$

Note the similarity to Diophantus here, see for example the translation in [1].

On page 205 is the rule \when multiplying powers, one sums the numbers of the abbreviations". Examples are given on pages 205 and 206. On page 207 root of powers are discussed and on page 208 the rule \when dividing a power and the two powers are equal or greater one subtracts the numbers of the abbreviations". Bombelli uses his notation in giving the rule. Bombelli does not possess negative exponents. What follows are many examples of working with polynomials.

On page 209 is some division notation, namely 20 ¥ 4x is written as

$$\frac{1}{4}\,\overline{)20} \text{ and as } \frac{20}{\frac{1}{4}}.$$

On page 211 are more poems telling of rules of calculation:

Sommare.
Piµ e piµ si aggionge e fa piµ
Meno, e meno si aggionge, e fa meno
Piµ , e meno si cava.
meno, e piµ si cava.
Sotrare.
Piµ di piµ si cava, e resta piµ , se quello disopra
è maggiore, ma se è minore, resta meno.
Meno di meno si cava, e resta meno, se è mag-
giore quel di sopra, ma fe è minore, resta piµ .
Meno di piµ si somma, e resta piµ .
Moltiplicare.
Piµ via piµ , fa piµ .
Meno via meno fa piµ .
Piµ via meno fa meno.
Meno via piµ fa meno.
E benche non ¯ ¯n dato regola nel
primo libro del partire, non di meno
perche in queste digniti potrebe
accadere: pero porro la fua regala.
Partire.
Piµ per piµ ne vien piµ .
Meno per meno ne vien piµ .
Meno per piµ ne vien meno.
Piµ per meno ne vien meno.

On pages 212 and 213 are columnar addition and subtraction, which was a surprise.

$$
\begin{array}{rl}
\text{Summa} & \overset{1}{6}\ .\text{p}.4. \\
\text{Con} & \overset{1}{5}\ .\text{p}.10. \\
\text{Fa} & \overset{1}{11}\ .\text{p}.14. \\
 & \\
\text{Di} & \overset{1}{4}\ .\text{p}.6. \\
\text{Causa} & \overset{1}{2}\ .\text{p}.5. \\
\text{Resta} & \overset{1}{2}\ .\text{p}.1.
\end{array}
$$

On page 214 is columnar multiplication, on page 216 is multiplication of polynomials, on page 224 Bombelli multiplies rational expressions which are quite sophisticated and on pages 224 and 225 he divides polynomials. Surprisingly, on page 229 Bombelli does long division on polynomials, and it appears almost exactly as we would write it (see Figure 2).

On page 231 he divides rational expressions by inverting the second expression and then multiplying. On page 240 he solves linear equations. On page 245 is a mean proportional diagram.

On page 247 Bombelli solves $x^3 = a$ and also $ax^3 = bx^2$. He begins solving quadratic equations (in all cases!) on page 248 ($2x^2 + 12x = 32$). Page 253 contains a geometric proof of solving quadratic equations (the usual diagram). More equations of quadratic type follow, for example $2x^4 + 12x^2 = 40$ (page 267) and $ax^6 + bx^3 = c$ (starting on page 275).

On page 280 we begin seeing cubic equations, $x^3 + ax = b$ and their solution using Cardano's formula. On page 281 Bombelli solves (without

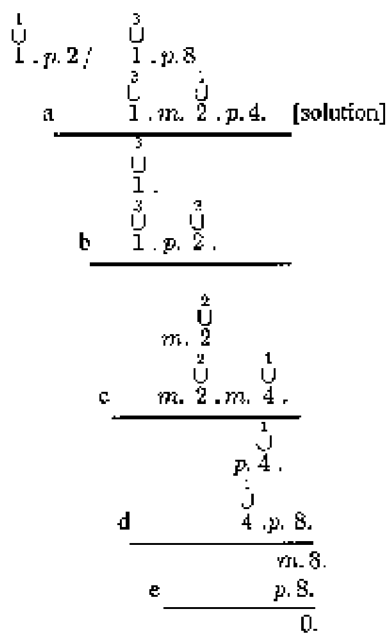

Figure 2:

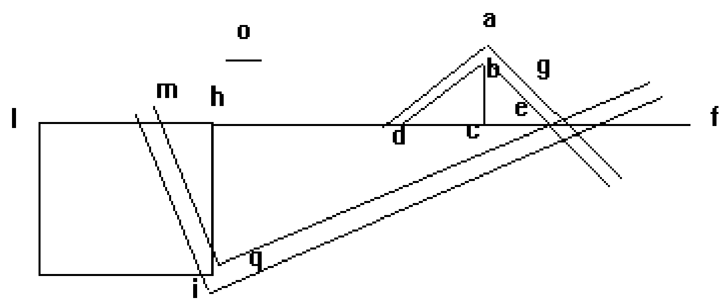

Figure 3:

much fanfare) $x^3 + 6x = 20$ yielding

R.c.bR.q.108.p.10c.p.R.c.bR.q.108.m.10c.

On page 285 is a geometric proof of Cardano's formula: Demonstracion dell Sorpradetto Capitulo di Cubo e Tanto eguale a numero. The example is $\overset{3}{[}$.p.$\overset{\overset{3}{[}}{6}$ eguale a 20 or $x^3 + 6x = 20$. The decomposition of the cube is given in a diagram similar to the picture on page 143 of [2].

On page 286, we see a \mechanical" device (Figure 3) for constructing a solution (not a curve) to the same equation $x^3 + 6x = 20$.

The square on lh has side $\sqrt{20}$ and the segment o has length 1. Extend lh until f. Make hc which is 6, and make dc which is 1. Take a perpendicular to c at a. Now Bombelli says \let us have two squadri materiale (carpenter's square?) which will be as can be seen in the diagram. We have g is a leg of the squadri materiale and the other is q and the one which is q, place right angle at i and the other part with the angle on the line ac with the arm across the line f." And so on. We are not sure what all this means.

On page 290 Bombelli discusses $x^3 = ax + b$ with the example $x^3 = 6x + 40$. Using Cardano's formula he gets

$$\begin{aligned} & \sqrt[3]{20 + \sqrt{392}} + \sqrt[3]{20 - \sqrt{392}} \\ = \; & 2 + \sqrt{2} + 2 - \sqrt{2} \\ = \; & 4. \end{aligned}$$

Bombelli does not verify the first equality, presumably because he has explained that type of calculation in Book I.

On pages 291 and 292 Bombelli solves $x^3 = 12x + 20$ and $x^3 = 12x + 9$, respectively. On page 293 we see the famous example, $x^3 = 15x + 4$, which upon using Cardano's formula one gets

$$x = \sqrt[3]{2 + 121\sqrt{-1}} + \sqrt[3]{2 - 121\sqrt{-1}}.$$

Bombelli then simply states that $x = 4$, again, presumably because he has explained that type of calculation in Book I. This is certainly not what one would expect after reading various history of mathematics books.

On page 298 is another construction with the squadri like above for $x^3 = 6x + 4$. On page 299 is "completing the cube", Transmutatione de Cubo eguale a Tanti, e numero in cubo, a potenzi eguale a numero. Bombelli then goes on to solve quartic equations of many types until the end of Book II.

Chapter 3, which goes from pages 416 to 650 (the end of the text), contains all problems and solutions, numbered up to problem 64.

References

[1] Ronald Calinger, Classics of Mathematics, Prentice Hall, 1995.

[2] William Dunham, Journey Throgh Genius: The Great Theorems of Mathematics, Wiley, 1990.

[3] John Fauvel and Jeremy Gray, The History of Mathematics, A Reader, The Open University, 1987.

[4] Victor Katz, A History of Mathematics: An Introduction, Harper Collins,1992.